# Complex Symmetric Matrices with Strongly Stable Iterates*

Eitan Tadmor
*School of Mathematical Sciences*
*Tel-Aviv University*
*Tel Aviv 69978, Israel*
*and*
*Institute for Computer Applications in Science and Engineering*
*NASA Langley Research Center*
*Hampton, Virginia 23665*

Submitted by Shmuel Friedland

ABSTRACT

We study complex-valued symmetric matrices. A simple expression for the spectral norm of such matrices is obtained, by utilizing a unitarily congruent invariant form. Consequently, we provide a sharp criterion for identifying those symmetric matrices whose spectral norm does not exceed one: such *strongly stable* matrices are usually sought in connection with convergent difference approximations to partial differential equations. As an example, we apply the derived criterion to conclude the strong stability of a Lax-Wendroff scheme.

## 1. INTRODUCTION

We study complex symmetric matrices, i.e., matrices $C$ whose entries $C_{jk}$ are complex-valued, and which coincide with their *real* transpose, $C_{jk} = C_{kj}$.

Such matrices arise, for example, as the amplification matrices associated with convergent difference approximations to (symmetric) partial differential equations: indeed, the stability question of the latter is governed by the power-boundedness of such complex symmetric amplification matrices $C$. In

*Research was supported in part by the National Aeronautics and Space Administration under NASA Contract No. NAS1-17070 while the author was in residence at ICASE, NASA Langley Research Center, Hampton, VA 23665. Additional support was provided in part by NSF Grant No. DMS85-03294 and ARO Grant No. DAAG29-85-K-0190 while in residence at UCLA, Los Angeles, CA 90024.




0024-3795/86/$3.50

1964 Lax and Wendroff [11] were first to utilize *numerical-radius* techniques in order to prove stability of their schemes by verifying

$$\operatorname*{Max}_{x^*x=1} |x^*Cx| \leqslant 1. \tag{1.1}$$

Halmos's inequality can be used to conclude that the powers of $C$ are then uniformly bounded by 2 (e.g. [4, 5, 9]). A stronger sufficient criterion for power-boundedness is provided by the inequality

$$\operatorname*{Max}_{x^*x=1} |x^*C^*Cx| \leqslant 1. \tag{1.2}$$

Indeed, by the submultiplicativity of the spectral norm, the matrix $C$ has strongly stable iterates in this case, all are uniformly bounded by 1. Such strongly stable schemes are usually sought in connection with problems admiting variable and nonlinear coefficients, splitting techniques, etc. (e.g. [1, 14, 17] and in particular [10]).

Unfortunately, calculating the spectral norm of a matrix may prove itself a complicated task, due to the quadratic appearance of $C$ on the right of (1.2). In the next section we recall the canonical Schur representation of such complex symmetric matrices, $C = C^t$, which yields a more favorable expression for their spectral norm,

$$\operatorname*{Max}_{x^*x=1} |x^tCx|.$$

The latter expression shares the advantage of the numerical radius in (1.1), namely, both involve *linear* form dependence on the matrix $C$. Consequently, we are able, in Section 3 below, to conclude with a sharp, relatively simple criterion for checking the strong stability of complex symmetric matrices, $C^*C \leqslant I$; specifically, in Section 4 it is recast into the requirement

$$(x^*Kx)(y^*Ky) \leqslant 2x^*Kx - (y^*Jx)^2, \qquad x^*x = y^*y = 1$$

where $-K$ and $J$ are respectively the real and imaginary parts of $C - I$. As an example, this criterion is then applied to prove the strong stability of the (modified) Lax-Wendroff scheme studied in [11].

## 2. SYMMETRY INVARIANCE UNDER UNITARY CONGRUENCE

Let $\mathbb{C}^n$ be the space of $n$-column complex vectors. Given a vector $x$ in $\mathbb{C}^n$, we let $\bar{x}$, $x^t$, and $x^* \equiv \bar{x}^t$ denote, respectively, the (complex) conjugate, the transpose, and the (complex) conjugate transpose of $x$. Similar notation is used for matrices.

Let $(x, y) = y^*x$ stand for the usual Euclidean inner product, and let $C$ be a given matrix in $M_n(\mathbb{C})$—the algebra of $n \times n$ complex-valued matrices. Among other quantities used to measure the size of a matrix $C$, we have its *spectral norm* —which will be temporarily denoted $N(C)$,

$$N(C) \equiv \operatorname*{Max}_{|x| = |y| = 1} |(Cx, y)|, \tag{2.1}$$

and its numerical and spectral radii, given respectively by

$$r(C) \equiv \operatorname*{Max}_{|x| = 1} |(Cx, x)|, \tag{2.2}$$

$$\rho(C) \equiv \operatorname*{Max}_{Cx = \lambda x,\ |x| = 1} |\lambda|. \tag{2.3}$$

Those three quantities admit the following hierarchy of inequalities, valid for all $C$ in $M_n(\mathbb{C})$:

$$\rho(C) \leqslant r(C) \leqslant N(C). \tag{2.4}$$

When does equality take place? In connection with this question one observes that (e.g. [4, 5, 9])

(1) equality holds for all *diagonal* matrices $\Lambda$:

$$\rho(\Lambda) = r(\Lambda) = N(\Lambda),$$

and

(2) each of the three quantities is invariant under *unitary similarities*; that is, for every unitary $U$, $U^*U = I_n$, and all $C$ in $M_n(\mathbb{C})$,

$$N(C) = N(U^*CU), \tag{2.5a}$$

$$r(C) = r(U^*CU), \tag{2.5b}$$

$$\rho(C) = \rho(U^*CU). \tag{2.5c}$$

As a consequence of the last two observations, equality in (2.4) follows for all matrices $C$ which are unitarily similar to diagonal ones, namely, *normal* matrices:

$$\rho(C) = r(C) = N(C), \qquad C^*C = CC^*. \tag{2.6}$$

In general, matrices satisfying the equality on the left of (2.6)—that is, equality between their spectral and numerical radii—are called *spectral matrices* after Halmos [9, p. 115]; such matrices were completely characterized in [6, 7]. Special cases are the *radial matrices* [9]—those having equal spectral radius and norm [2, 8, 13]. According to this terminology, we have seen that the class of normal and, in particular *real* symmetric matrices is contained in the radial class; indeed, it is a proper subclass of the latter [4, 8].

Yet, the class of *complex* symmetric matrices, which we are interested in here, is included in none of the above. This is essentially due to the fact that this class is not invariant under unitary similarities. Rather, the symmetry of (complex-valued) matrices is invariant under (transposed-type) congruence: if $C$ coincides with its transpose, so does $U^tCU$. This motivate our discussion below, regarding the slightly different analogue quantities of what we had before, which are more adequate for our purposes of studying complex symmetric matrices.

To begin with, we introduce for an arbitrary matrix $C$ in $M_n(\mathbb{C})$, the associated congruent-type quantities, namely, the *congruent-type norm*, $\mathrm{Nc}(C)$,

$$\mathrm{Nc}(C) \equiv \operatorname*{Max}_{|x| = |y| = 1} |(Cx, \bar{y})|, \tag{2.7}$$

and the *congruent-type numerical* and *spectral radii*, given respectively by

$$\mathrm{rc}(C) \equiv \operatorname*{Max}_{|x| = 1} |(Cx, \bar{x})|, \tag{2.8}$$

$$\rho\mathrm{c}(C) \equiv \operatorname*{Max}_{Cx = \lambda\bar{x},\ |x| = 1} |\lambda|. \tag{2.9}$$

As before, we have the analogue hierarchy of inequalities, valued for all matrices $C$ in $M_n(\mathbb{C})$,

$$\rho\mathrm{c}(C) \leqslant \mathrm{rc}(C) \leqslant \mathrm{Nc}(C). \tag{2.10}$$

Seeking equality in (2.10), rather standard arguments, which we omit, tell us that

(1) equality holds for all *diagonal* matrices $\Lambda$:

$$\rho c(\Lambda) = rc(\Lambda) = Nc(\Lambda),$$

and, at the heart of the matter,

(2) each of the three (congruent-type) quantities is invariant under *unitary congruence*; that is, for every unitary $U$, $U^*U = I_n$ and all $C$ in $M_n(\mathbb{C})$

$$Nc(C) = Nc(U^tCU), \tag{2.11a}$$

$$rc(C) = rc(U^tCU), \tag{2.11b}$$

$$\rho c(C) = \rho c(U^tCU). \tag{2.11c}$$

Hence, equality in (2.10) follows for all matrices $C$ which are unitarily congruent to diagonal ones: a classical result of Schur [15, 16] asserts that these are exactly the (possibly complex-valued) symmetric matrix. We state our conclusion as

LEMMA 2.1 (e.g. [12, Lemma 3.7]). *Let $C$ be a complex-valued symmetric matrix, $C = C^t$. Then we have*

$$\rho c(C) = rc(C) = Nc(C). \tag{2.12}$$

Several remarks are in order:

(1) Since the conjugate of a unit vector is another unit vector, the spectral norm $N(\cdot)$ and its congruent-type analogue $Nc(\cdot)$ coincide. Both will be denoted below, as customary, by $\|\cdot\|$:

$$N(C) = Nc(C) = \|C\| \equiv \operatorname*{Max}_{|x|=1} |Cx|. \tag{2.13}$$

Thus, the right-hand equality stated in Lemma 2.1 reads $rc(C) = \|C\|$, or, written explicitly

$$\operatorname*{Max}_{|x|=1} |(Cx, \bar{x})| = \operatorname*{Max}_{|x|=1} |Cx|, \qquad C = C^t. \tag{2.14}$$

(2) Let $x$ be a particular vector at which the maximum on the left of (2.14) is attained. Then the equality asserted in (2.14) is a special case of the

Cauchy-Schwarz inequality

$$|(Cx, \bar{x})| \leqslant |Cx| \cdot |\bar{x}|, \qquad |x| = 1.$$

This, in turn, implies that the vectors $Cx$ and $\bar{x}$ are parallel; the vector $x$ is therefore necessarily a congruent-type eigenvector corresponding to a congruent-type eigenvalue $\lambda$,

$$Cx = \lambda \bar{x},$$

such that $|\lambda| = \rho c(C) = rc(C)$. Hence, we obtain an independent derivation of the left-hand equality stated in Lemma 2.1, which follows directly from the corresponding right-hand one. We shall refer to such $\lambda$ lying on the circle $|z| = \rho c(C)$ as a congruent-type spectral eigenvalue.

(3) Once the existence of a congruent-type spectral eigenvalue has been established, a different derivation of Lemma 2.1 can be given. For, if $\lambda$ is a congruent-type spectral eigenvalue satisfying $Cx = \lambda \bar{x}$, then by the symmetry of $C$, $\bar{\lambda}$ is a congruent-type one for $C^*$, $C^* \bar{x} = \bar{\lambda} x$; hence $C^*Cx = |\lambda|^2 x$, and therefore $|\lambda|^2$ equals $\rho(C^*C) \equiv \|C\|^2$. Thus, we have shown that $\rho c(C) = \|C\|$ and (2.12) follows. Indeed, the congruent-type eigenvalues of $C$ are exactly the principal values of that matrix—they are uniquely determined up to multiplication by a unit scalar.

(4) In [17], Turkel has shown that in order to calculate the numerical radius of a complex symmetric matrix $C$, it is enough to maximize the form $|(Cx, x)|$ over the *real* unit ball; we may therefore write the numerical radius of such a matrix as

$$r(C) = \operatorname*{Max}_{x \in \mathbf{R}^n,\ |x| = 1} |(Cx, \bar{x})|, \tag{2.15}$$

while according to Lemma 2.1, the spectral norm is obtained by an extension of the (complex-valued) unit ball,

$$\|C\| = \operatorname*{Max}_{x \in \mathbf{C}^n,\ |x| = 1} |(Cx, \bar{x})|. \tag{2.16}$$

## 3. THE SPECTRAL NORM OF SYMMETRIC MATRICES

The calculation of a matrix spectral norm,

$$\|C\| = \operatorname*{Max}_{|x| = 1} |(Cx, Cx)|^{1/2},$$

may prove a complicated task due to the quadratic appearance of the matrix $C$ on the right. In the symmetric case, Lemma 2.1 allows us, instead, to calculate the simpler congruent-type numerical radius

$$\mathrm{rc}(C) = \underset{|x|=1}{\mathrm{Max}} |(Cx, \bar{x})|.$$

The advantage of the latter lies in its simple, linear dependence on $C$, similar to that of the numerical radius

$$r(C) = \underset{|x|=1}{\mathrm{Max}} |(Cx, x)|.$$

In Section 4 we shall make use of this advantage, while verifying the strong stability of certain Lax-Wendroff difference approximations. To this end we first prepare the following proposition, putting Lemma 2.1 in a more convenient form.

LEMMA 3.1. *Let $C = R + iJ$ be a symmetric matrix with $R$ and $J$ denoting respectively its real and imaginary parts. We then have*

$$\|C\| = \underset{|u|^2+|v|^2=1}{\mathrm{Max}} \left[ (Ru, u) + 2|(Ju, v)| - (Rv, v) \right]. \tag{3.1}$$

*Proof.* Since $C$ is a symmetric matrix, then by Lemma 2.1 its spectral norm equals its congruent-type spectral radius:

$$\|C\| = \rho c(C). \tag{3.2}$$

Turning to calculate the latter, we first observe that congruent-type eigenvalues are determined up to multiplication by a unit scalar; indeed, the following equality is $\theta$-independent:

$$Cx(\theta) = \lambda e^{2i\theta}\overline{x(\theta)}, \quad x(\theta) \equiv xe^{i\theta}, \qquad 0 \leqslant \theta \leqslant 2\pi.$$

Let $\lambda$ be a congruent-type spectral eigenvalue, which is assumed—without loss of generality—to be real:

$$\lambda = \rho c(C). \tag{3.3}$$

If $x = u + iv$ is the corresponding congruent-type eigenvector,

$$(R + iJ)(u + iv) = \lambda(u - iv), \tag{3.4a}$$

then equating real and imaginary parts yields

$$\begin{bmatrix} R & -J \\ -J & -R \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} u \\ v \end{bmatrix}. \tag{3.4b}$$

Hence, $\lambda$ is a spectral eigenvalue of the real symmetric matrix on the left, $\mathscr{C}$:

$$\lambda = \rho(\mathscr{C}), \qquad \mathscr{C} = \begin{bmatrix} R & -J \\ -J & -R \end{bmatrix}. \tag{3.5}$$

Furthermore, since $\mathscr{C}$ is a real symmetric matrix, then according to (2.6) it is, in particular, a spectral one, i.e.,

$$\rho(\mathscr{C}) = r(\mathscr{C}). \tag{3.6}$$

The equalities (3.2), (3.3), (3.5), and (3.6) imply

$$\|C\| = r(\mathscr{C}) \equiv \operatorname*{Max}_{|u|^2 + |v|^2 = 1} |(Ru, u) - 2(Ju, v) - (Rv, v)|;$$

choosing the sign of $\pm v$ so that $-(Ju, v) = |(Ju, v)|$ and exchanging $u$ and $\pm v$ if necessary, so that $(Ru, u) > (Rv, v)$, the lemma follows. ∎

We remark that Lemma 3.1 can be generalized, formulating its conclusion in a more symmetric fashion. To this end, let us replace $\lambda$ in (3.4a) with the congruent-type spectral eigenvalue $\lambda e^{i\theta}$, $0 \leqslant \theta \leqslant 2\pi$. The same arguments detailed above lead to the equality

$$\|C\| = r(\mathscr{C}), \qquad \mathscr{C} = \mathscr{C}(\theta) = \begin{bmatrix} R(\theta) & -J(\theta) \\ -J(\theta) & -R(\theta) \end{bmatrix}, \tag{3.7a}$$

where $R(\theta)$ and $J(\theta)$ are given by

$$R(\theta) = \cos\theta R + \sin\theta J, \qquad J(\theta) = \cos\theta J - \sin\theta R. \tag{3.7b}$$

Consequently, the matrices $R$ and $J$ appearing on the right-hand side of (3.1) should be replaced with $R(\theta)$ and $J(\theta)$ respectively, yielding for arbitrary $\theta$, $0 \leqslant \theta \leqslant 2\pi$,

$$\|C\| = \operatorname*{Max}_{|u|^2 + |v|^2 = 1} \big|\cos\theta\,[(Ru, u) + 2(Ju, v) - (Rv, v)] + \sin\theta\,[(Ju, u) - 2(Ru, v) - (Jv, v)]\big|. \tag{3.8}$$

Lemma 3.1 refers to the special case $\theta = 0$.

Using Lemma 3.1, we conclude with

COROLLARY 3.2. *Let $C = R + iJ$ be a symmetric matrix with $R$ and $J$ denoting respectively its real and imaginary parts. We then have*

$$\|C\| \leqslant \frac{1}{2} \operatorname*{Max}_{|x|=|y|=1} \left[ (Rx,x) - (Ry,y) + \sqrt{[(Rx,x)+(Ry,y)]^2 + 4(Jx,y)^2} \right]. \tag{3.9}$$

*Proof.* According to Lemma 3.1, the spectral norm of $C = R + iJ$ is given by a maximal combination of the form

$$(Ru,u) + 2|(Ju,v)| - (Rv,v), \qquad |u|^2 + |v|^2 = 1. \tag{3.10}$$

We rewrite (3.10) in the following way:

$$(Rx,x)\sin^2\phi + 2|(Jx,y)|\sin\phi\cos\phi - (Ry,y)\cos^2\phi; \tag{3.11}$$

here $x$ and $y$ are the normalized unit vectors $x = u/|u|$ and $y = v/|v|$ with $\sin\phi = |u|$, $\cos\phi = |v|$ whose squares sum to one.

The result follows by computing the extremum of the expression (3.11) with respect to the argument $\phi$. ■

## 4. STRONGLY STABLE SYMMETRIC MATRICES

In this section we examine symmetric matrices whose spectral norm does not exceed one; such *strongly stable* matrices are usually sought in connection with convergent difference approximations to partial differential equations. As an example, we shall utilize our results to conclude the strong stability of a certain Lax-Wendroff scheme.

To begin with, we state the following sufficiency criterion.

LEMMA 4.1. *Let $C = R + iJ$ be a symmetric matrix with $R$ and $J$ denoting respectively its real and imaginary parts. Then $C$ is strongly stable, $\|C\| \leqslant 1$, provided*

$$(Rx,x)^2 + (Jx,y)^2 \leqslant 1 - [(Rx,x) - (Ry,y)][1 - (Rx,x)] \tag{4.1}$$

*for all unit vectors $|x| = |y| = 1$.*

*Proof.* According to Corollary 3.2, strong stability follows if the inequality

$$\tfrac{1}{2}\sqrt{[(Rx,x)+(Ry,y)]^2+4(Jx,y)^2} \leqslant 1-\tfrac{1}{2}[(Rx,x)-(Ry,y)] \tag{4.2}$$

holds for all unit vectors $|x|=|y|=1$; see (3.9). By choosing $x=y$ our assumption in (4.1) implies, in particular, that $\rho(R)\leqslant 1$. Hence, the right-hand side of (4.2) is nonnegative and the result follows by squaring both of its sides. ∎

REMARK 4.2. It is instructive at this point to compare the last strong stability criterion, with the requirement

$$r(C)\leqslant 1, \tag{4.3a}$$

which was originally used as a stability criterion by Lax and Wendroff in [11]. Setting $K=I-R$, the requirement (4.3a) for a symmetric matrix, $C=C^t$, reads [11]

$$(Kx,x)^2\leqslant 2(Kx,x)-(Jx,x)^2, \qquad |x|=1, \tag{4.3b}$$

while for strong stability,

$$\|C\|\leqslant 1, \tag{4.4a}$$

we need—according to Lemma 4.1—the slightly stronger

$$(Kx,x)(Ky,y)\leqslant 2(Kx,x)-(Jx,y)^2, \qquad |x|=|y|=1. \tag{4.4b}$$

For a later purpose, we shall quote here an immediate corollary of the strong stability criterion (4.4b), stating

COROLLARY 4.3. *Let $C=I-K+iJ$ be a symmetric matrix. Then $C$ is strongly stable,* $\|C\|\leqslant 1$, *provided*

$$(Kx,x)(Ky,y)\leqslant (2K-J^2x,x), \qquad |x|=|y|=1. \tag{4.5}$$

The corollary follows upon employing Cauchy-Schwarz inequality to the last term on the right of (4.4b), yielding

$$(2K - J^2x, x) \leqslant 2(Kx, x) - (Jx, y)^2.$$

In the rest of this section we utilize Corollary 4.3 to verify the strong stability of a certain (modified) Lax-Wendroff scheme [11]. The problem is governed by the strong stability of a so-called amplification matrix given by

$$C = C(\xi, \eta) = I - K + iJ; \tag{4.6a}$$

here $K$ and $J$ are polynomials in the real symmetric matrices $A$ and $B$, which take the form

$$J = J(\xi, \eta) = \sin\xi \cdot \lambda A + \sin\eta \cdot \mu B, \tag{4.6b}$$

$$K = K(\xi, \eta) = \tfrac{1}{2}\left[(\alpha + \beta)(\alpha\lambda^2A^2 + \beta\mu^2B^2) + J^2\right],$$

$$\alpha \equiv 1 - \cos\xi, \quad \beta \equiv 1 - \cos\eta. \tag{4.6c}$$

Our purpose is to show that for sufficiently small scalars $\lambda$ and $\mu$, the amplification matrix $C(\xi, \eta)$ in (4.6) is strongly stable for all $\xi, \eta$, $0 \leqslant \xi, \eta \leqslant 2\pi$.

Using the abbreviations

$$a \equiv a(x) = \lambda|Ax|, \qquad b \equiv b(x) = \mu|Bx|, \qquad |x| = 1,$$

we find

$$(J^2x, x) \leqslant \sin^2\xi \cdot a^2 + \sin^2\eta \cdot b^2 + 2\sin\xi \cdot \sin\eta \cdot ab,$$

with the last term on the right not exceeding a value of

$$2\sin\xi \cdot \sin\eta \cdot ab \leqslant (1 - \cos\xi)(1 + \cos\eta)a^2 + (1 - \cos\eta)(1 + \cos\xi)b^2.$$

Inserted into (4.6c), we arrive at the essential estimate

$$\begin{aligned}(Kx, x) &\leqslant \tfrac{1}{2}\big[(\alpha + \beta)(\alpha a^2 + \beta b^2) + \alpha(2 - \alpha)a^2 \\ &\qquad + \beta(2 - \beta)b^2 + \alpha(2 - \beta)a^2 + \beta(2 - \alpha)b^2\big] \\ &= 2(\alpha a^2 + \beta b^2).\end{aligned} \tag{4.7}$$

In their original treatment, Lax and Wendroff employed a somewhat different estimate of the same term [11, p. 392],

$$(Ky, y) \leqslant (\sqrt{\alpha} + \sqrt{\beta})\left[\sqrt{\alpha}\, a^2(y) + \sqrt{\beta}\, b^2(y)\right],$$

which yields

$$(Ky, y) \leqslant \sqrt{2}\,(\alpha + \beta)\left[a^4(y) + b^4(y)\right]^{1/2}. \tag{4.8}$$

The last two estimates provide us with the necessary upper bounds on the two terms appearing on the left of (4.5); regarding the right-hand side of (4.5), we have in view of (4.6c)

$$(2K - J^2x, x) = (\alpha + \beta)(\alpha a^2 + \beta b^2). \tag{4.9}$$

Hence, Corollary 4.3 yields strong stability provided the inequality

$$\left[\lambda^4 |Ay|^4 + \mu^4 |By|^4\right]^{1/2} \leqslant \frac{(\alpha + \beta)(\alpha a^2 + \beta b^2)}{\sqrt{2} \cdot 2(\alpha + \beta)(\alpha a^2 + \beta b^2)}$$

holds for all unit vectors $|y| = 1$.

We summarize what we have shown in

THEOREM 4.4. *The Lax-Wendroff scheme* (4.6) *is strongly stable provided the so-called CFL condition is fulfilled*:

$$8(\lambda^4 A^4 + \mu^4 B^4) \leqslant I. \tag{4.10}$$

The strong stability condition derived in (4.10) turns out to yield a slight improvement over the strong stability condition obtained for this case by Abarbanel and Gottlieb in [1], requiring

$$4 \max(\lambda^2 A^2, \mu^2 B^2) \leqslant I.$$

The two conditions coincide whenever $\lambda A = \mu B$, in which case they agree with the somewhat more permissive Lax-Wendroff condition [11, Theorem 4.4] requiring $2(\lambda^2 A^2 + \mu^2 B^2) \leqslant I$. The point we make here is that our general algebraic criteria for strong stability—consisting of Lemma 4.1 and its stricter version in Corollary 4.3—are both sharp enough for the purpose of studying the stability question in a rather systematic way, replacing the brute-force proof employed in [1].

*I would like to thank S. Friedland for bringing to my attention Reference [12].*

## REFERENCES


1 S. Abarbanel and D. Gottlieb, On proving stability of multidimensional difference schemes, *Linear Algebra Appl.* 11:247–250 (1975).
2 H. Flanders, On the norm and spectral radius, *Linear and Multilinear Algebra* 2:239–240 (1974).
3 S. Friedland and E. Tadmor, Optimality of the Lax-Wendroff condition, *Linear Algebra Appl.* 56:121–129 (1984).
4 M. Goldberg, On certain finite dimensional numerical ranges and numerical radii, *Linear and Multilinear Algebra* 7:329–342 (1979).
5 M. Goldberg and E. Tadmor, On the numerical radius and its applications, *Linear Algebra Appl.* 42:263–284 (1982).
6 M. Goldberg, E. Tadmor, and G. Zwas, The numerical radius and spectral matrices, *Linear and Multilinear Algebra* 2:317–326 (1975).
7 M. Goldberg, E. Tadmor, and G. Zwas, Numerical radius of positive matrices, *Linear Algebra Appl.* 12:209–214 (1975).
8 M. Goldberg and G. Zwas, On matrices having equal spectral radius and spectral norms, *Linear Algebra Appl.* 8:427–434 (1974).
9 P. R. Halmos, *A Hilbert Space Problem Book*, Van Nostrand, New York, 1967.
10 P. D. Lax and L. Nirenberg, On stability for difference schemes; a sharp form of Gårding's inequality, *Comm. Pure Appl. Math.* 19:473–492 (1966).
11 P. D. Lax and B. Wendroff, Difference schemes for hyperbolic equations with high order of accuracy, *Comm. Pure Appl. Math.* 17:381–391 (1964).
12 C. Pommerenke, *Univalent Functions*, Vandenhoeck and Ruprecht, Göttingen, 1975.
13 V. Pták, Isometric parts of an operator and the critical exponent, *Časopis Pěst. Mat.* 101:383–388 (1976).
14 R. D. Richtmyer and D. W. Morton, *Difference Methods for Initial-Value Problems*, 2nd ed., Interscience, New York, 1967.
15 I. Schur, Bemerkungen zur Theorie der Beschränkten Bilinearformen mit unendlich vielen Veränderlichen, *J. Reine Angew. Math.* 140:1–28 (1911).
16 I. Schur, Ein Satz über quadratische Formen mit komplexen Koeffizienten, *Amer. J. Math.* 67:472–480 (1945).
17 E. Turkel, Symmetric hyperbolic difference schemes and matrix problems, *Linear Algebra Appl.* 16:109–129 (1977).


*Received 14 February 1985; revised 20 June 1985*